Received: *February 9, 1998*, Accepted: *July 27, 1998*

Spinor-type objects in metric-affine space-time

Eugene A. Poberii[*]

A.A. Friedmann Laboratory for Theoretical Physics,
Griboedov Can. 30/32, St. Petersburg 191023, Russia
E-mail: poberii@chat.ru

Abstract: A generalized spinor structure is constructed as a $GL(2,\mathbb{C})$ principal fiber bundle over the metric-affine manifold. Relations between vectors and spinors are established explicitly through the generalized Pauli and Dirac matrices. An expression for generalized spinor connection is obtained. The Dirac equation in metric-affine space-time is written in the form representing an interaction of spinors with nonmetricity explicitly. A possible physical meaning of nonmetrical objects is discussed in brief.

Keywords: Classical Theories of Gravity, Differential and Algebraic Geometry.

[*]The author acknowledges the support of RFBR under the grant No 96-02-19528.

Contents

1. Introduction

Recently we have proposed a scale-covariant gravitational theory [1, 2] based on the complete relaxing of Riemannian constraints (which force the affine connection to be symmetric and metric compatible) combined with the requirement of local conformal invariance (which excludes the pre-existence of an intrinsic scale). The resulting theory admits an independent coexistence of metric and affine structures on the manifold and does not contain any fixed unit of measurement. As we have shown [1], a preferable system of units could be arisen in the result of an appropriate spontaneous gauge fixing but the existence of a preferable Riemannian structure on the space-time manifold does not follow from the first physical principles. The Riemann-Cartan structure could be arisen in the result of inflationary evolution of the Universe [2] since the nonmetrical properties of space-time could be responsible for the inflationary stage. However the above approach was incomplete for it did not contain a consistent description of spinor fields in arbitrary metric-affine space-time. In this paper we shall try to fill this gap and incorporate spinorial objects into the theory.

As is known, spinors were introduced by E.Cartan [3] through a remarkable construction (Cartan's map) in which the components of an isotropic vector satisfying the equation of isotropic cone were represented through the Hermitean combination of components of a complex two-component vector (called spinor). Moreover the two-component Cartan spinors are the elements of the space of fundamental spinorial representation of the Lorentz group. This is a remarkable feature since in special relativity our definition of a "particle" is based on the irreducible unitary representations of the Poincaré group, which is a semidirect product of translations T and the Lorentz group $SO(1,3)$. The double-valued representations of $SO(1,3)$ acting on two-dimensional complex vector space are called spinor representations and describe particles with half integer spin. Usual spinors therefore possess two main features in Minkowski space-time: (i) there is a simple one-to-one correspondence between Hermitean second-rank spinors and four-vectors, and (ii) spinors as objects of two-dimensional complex space carry double valued representations of the Lorentz group. These two features both can be preserved in Riemannian space-time but not in metric-affine space-time.

In Riemannian space-time spinors in Cartan's sense were first introduced by Fock and Iwanenko [4] by constructing a spinor equivalence of the operation of parallel transport of the orthonormal tetrad. The conventional spinor formalism [4]—[11] in curved space-time has been developed within the framework of metric theories, i.e. under the Riemannian constraint $\nabla_\lambda g_{\mu\nu} = 0$. Thus it cannot be applied to the metric-affine gravity and must be generalized. Such a generalization can be done in two different ways.

When going from flat Minkowski space-time to the metric-affine, one needs to enlarge the group of transformation of tangent space to general linear group $GL(4,\mathbb{R})$ in which the Lorentz group is embedded. However, it turns out to be that $GL(4,\mathbb{R})$ does not admit finite-dimensional representations which on the orthogonal subgroup coincide with spinorial ones. In other words, the usual spinor representation of the Lorentz group cannot be expanded to the linear finite-dimensional representation of the general linear group. If one wishes to preserve the ordinary notion of spinors as objects realizing double-valued representations of the Lorentz group, then one is inevitably led to infinite-dimensional representations of the general linear group $GL(4,\mathbb{R})$. Such a generalization of spinors on the metric-affine manifold has been proposed by Ne'eman [12]. Spinor fields in this approach are represented as "manifields" carrying infinite-dimensional representations of the orthogonal subgroup (for comprehensive review of this approach see Ref. [13] and references therein). However in this way we lose the simple clear correspondence between vectors and spinors and should consider vector fields as infinite-dimensional as well. Moreover up to now no dynamic scheme was found for such manifields, i.e. no Lagrangian and no equations of motion. Within this approach it is unclear how the ordinary fermionic matter can be recovered from the manifields when nonmetricity vanishes.

Another way of introducing spinors into metric-affine space-time is to consider them simply as objects of two-dimensional complex vector space. The special linear group of unimodular complex 2×2 matrices, $SL(2,\mathbb{C})$, acting in this space is a universal covering of the Lorentz group $SO(1,3)$. In this way Geroch [10] introduced a spinor structure as an $SL(2,\mathbb{C})$ principal fiber bundle over a general space-time manifold with locally

Lorentz metric. According to Geroch's theorem [10] the existence of the spinor $SL(2,\mathbb{C})$ structure on the manifold implies that there exists a continuous orthonormal tetrad on this manifold. Following this point of view Adler [15] developed a spinor theory in the Weyl manifold. Hayashi and Shirafuji [16], introducing the spinor structure on the metric-affine space-time, have shown that the most general manifold admitting spinors in Cartan's sense, i.e. as objects carrying double-valued representations of the Lorentz group, is Weyl-Cartan. Therefore it has been shown that Cartan's spinors admit the enlargement of the structural group of standard spinor bundle up to a direct product $SL(2,\mathbb{C}) \otimes GL(1,\mathbb{C})$ where $GL(1,\mathbb{C})$ is the dilatation group in spinor space. The corresponding bundle of conformal spinorial frames double covers the bundle of conformal linear vector frames with the structural group in the form of direct product $SO(1,3) \otimes GL(1,\mathbb{R})$ where $GL(1,\mathbb{R})$ the dilatation group in tangent vector space. The possible generalization is to enlarge the structure group of this bundle to $GL(2,\mathbb{C})$. This step also implies a modification of the usual notion of a spinor. In support of this approach one can put forward two essential observations:

(i) All the conception of elementary particles depends on the existence and on the structure of the group of motions. In flat Minkowski space-time this group coincides with the Poincaré group. Thus here we have a possibility to classify particles by their masses and spins which are the eigenvalues of the Casimir operators of the Poincaré group. But what will happen with the notion of "particle" in curved space-time where the Poincaré group has only local sense and when this space-time admits a global group of motions different from the Poincaré group? As has been shown [14] even in Riemannian space-time the concept of "particle" is not a well defined notion. It has any definite sense only locally.

(ii) In Riemannian space-time spinor fields are usually introduced at every point as objects with components transforming according to spinor (double valued) representation of the orthogonal group of transformations of the orthonormal bases (tetrads) of the tangent space at this point. The main drawback of this definition of spinor fields in curved space-time is that the choice of a system of orthonormal tetrads cannot be fixed in a unique way without additional conditions. Moreover this choice is not connected with the geometry of space-time and is an additional artificial construction.

Therefore there is no need to relate "world" fields with representations of the Poincaré group and in particular there is no need to relate "world" spinors with transformations of the orthonormal tetrad. It is sufficient to regard them only as objects of two-component complex vector space. In this paper we shall try to clarify this problem and construct a generalized spinor structure as a $GL(2,\mathbb{C})$ principal fiber bundle over the metric-affine space-time manifold.

The paper is organized as follows. In Sec. 2 we consider a generalized two-component spinor algebra at a single point of the space-time manifold and establish relations between the complex spinor space and tangent vector space. In Sec. 3 we pass from two-component spinors to four-component spinors and put into consideration generalized γ-matrices. In Sec. 4 we complete the construction of $GL(2,\mathbb{C})$ bundle by introducing the rule for parallel transport of the fibers. Here we obtain the expression for the spinor connection and establish relations between nonmetricity and parallel transport of spinor metric. We end up with the Dirac equation in Sec. 5. In the last section we discuss some obtained results.

2. Generalized Two-Component Spinor Analysis at a Single Spacetime Point

We assume that the spacetime continuum M is a Hausdorff, connected, C^∞ four-dimensional manifold with a linear affine connection and a symmetric metric tensor $\boldsymbol{g}$ of Lorentz signature $(+---)$. No constraints on the metric and connection are imposed, so that affine and metric structures of the manifold are completely independent. We call this manifold a metric-affine space-time and denote it by (L_4, g). The manifold M is assumed also to possess an antisymmetric spinor metric $\boldsymbol{\varepsilon}$ and spinor affine connection.

We shall try to endow such a manifold with a generalized spinor structure, i.e. we shall construct a $GL(2,\mathbb{C})$ principal fiber bundle over M. For this goal we should study the structure of the underlying bundle of linear frames over M and make it consistent with the spinor structure.

The bundle of linear frames (co-frames) over (L_4, g) is the *principal fiber bundle* with the structure group $GL(4,\mathbb{R})$. The metric tensor $\boldsymbol{g}$ and its inverse $\boldsymbol{g}^{-1}$ give the following maps between tangent and cotangent spaces at a point x:

$$T_x(M)\times T_x(M)\xrightarrow{\boldsymbol{g}}\mathbb{R}\,;\qquad T_x(M)\xrightarrow{\boldsymbol{g}}T^*_x(M)\,; \tag{2.1}$$

$$T^*_x(M)\times T^*_x(M)\xrightarrow{\boldsymbol{g}^{-1}}\mathbb{R};\qquad T^*_x(M)\xrightarrow{\boldsymbol{g}^{-1}}T_x(M)\,. \tag{2.2}$$

An abstract spinor bundle can be constructed *independently* of the existence of vector structure of the manifold in the following way. Let us attach at each point x of the manifold M a two-dimensional complex vector space $S_x(M)=\mathbb{C}_2$ with the antisymmetric metric $\boldsymbol{\varepsilon}$. We call the elements of $S_x(M)$ *two-component spinors.* Let $S^*_x(M)$ be the space dual to $S_x(M)$. One can construct an abstract *spinor bundle* $\boldsymbol{S}(M)$ (*co-spinor bundle* $\boldsymbol{S}^*(M)$) as a disconnected collection of all $S_x(M)$ ($S^*_x(M)$)at each point of M:

$$\boldsymbol{S}(M)=\bigcup_{x\in M}S_x(M);\qquad \boldsymbol{S}^*(M)=\bigcup_{x\in M}S^*_x(M)\,. \tag{2.3}$$

Let $\boldsymbol{\omega}=\{\boldsymbol{\omega}_A\}(A=1,2)$ be a basis of $S_x(M)$ called the *spinor frame*, $\boldsymbol{\omega}^*=\{\boldsymbol{\omega}^A\}$ be a basis of $S^*_x(M)$ (*spinor co-frame*).Hereafter capital Latin indices take values 1,2. Let $G_x(M)$ be a set of all spinor bases of $S_x(M)$, and $G^*(M)$ be a set of all spinor bases of $S^*_x(M)$. Then the *bundle of linear spinor frames (co-frames)* is a collection of all spinor frames (co-frames) over M:

$$\boldsymbol{G}(M)=\bigcup_{x\in M}G_x(M);\qquad \boldsymbol{G}^*(M)=\bigcup_{x\in M}G^*_x(M)\,. \tag{2.4}$$

Different bases in $S_x(M)$ are connected by a linear transformation

$$\boldsymbol{\omega}'_B = G^A_B \boldsymbol{\omega}_A\,; \qquad G^A_B \in GL(2,\mathbb{C})\,, \tag{2.5}$$

whereas those in $S^*_x(M)$ are connected by the inverse transformations

$$\boldsymbol{\omega}^{A'} = (G^A_B)^{-1}\boldsymbol{\omega}^B\,, \tag{2.6}$$

so that the bundle of spinor frames (co-frames) is the principal fiber bundle with the structural group $GL(2,\mathbb{C})$.

We assume the existence of the antisymmetric metric spinor $\boldsymbol{\varepsilon}$ and its inverse $\boldsymbol{\varepsilon}^{-1}$ that give the following maps at a given point x:

$$S_x(M)\times S_x(M) \xrightarrow{\boldsymbol{\varepsilon}} \mathbb{C}\,; \quad S_x(M) \xrightarrow{\boldsymbol{\varepsilon}} S^*_x(M)\,; \tag{2.7}$$

$$S^*_x(M)\times S^*_x(M) \xrightarrow{\boldsymbol{\varepsilon}^{-1}} \mathbb{C}\,; \quad S^*_x(M) \xrightarrow{\boldsymbol{\varepsilon}^{-1}} S_x(M)\,. \tag{2.8}$$

Since $S_x(M)$ and $S^*_x(M)$ are complex spaces one should also take into consideration their complex-conjugates $\bar{S}_x(M)$ and $\bar{S}^*_x(M)$. The elements of these spaces are usually called *dotted spinors* for the reason that their spinor indices are provided by a dot above the index letter.

One can readily construct the bundle of higher rank two-component spinors as follows. From the spaces $S_x(M)$, $S^*_x(M), \bar{S}_x(M)$ and $\bar{S}^*_x(M)$ one can construct the tensor product:

$$\begin{aligned}\left(S^{p\dot{q}}_{r\dot{s}}\right)_x(M) \equiv\ & \underbrace{S_x(M)\otimes\cdots\otimes S_x(M)}_{p\ \text{times}}\otimes\underbrace{S^*_x(M)\otimes\cdots\otimes S^*_x(M)}_{q\ \text{times}} \\ & \otimes\underbrace{\bar{S}_x(M)\otimes\cdots\otimes \bar{S}_x(M)}_{\dot{r}\ \text{times}}\otimes\underbrace{\bar{S}^*_x(M)\otimes\cdots\otimes \bar{S}^*_x(M)}_{\dot{s}\ \text{times}}\,.\end{aligned} \tag{2.9}$$

We call an element of $\left(S^{p\dot{q}}_{r\dot{s}}\right)_x(M)$ a spinor of type $(p,q,\dot{r},\dot{s})$ which is p-times contravariant, q-times covariant and $\dot{r}$-times dotted contravariant, $\dot{s}$-times dotted covariant.

So far the constructions of tangent bundle and spinor bundle were completely independent. But the remarkable feature of two-component spinors is that they are related directly to vectors of the tangent space. Here we establish this relationship explicitly.

Let us consider now at a given point $x\in M$ the Hermitean second-rank spinors which form the space $S_x(M)\underset{H}{\otimes}\bar{S}_x(M)$. Here the subscript H means Hermitization. Any Hermitean second-rank spinor $\chi^{A\dot{B}}\in S_x(M)\underset{H}{\otimes}\bar{S}_x(M)$

$$\chi^{A\dot{B}} = \chi^{\dot{B}A} \tag{2.10}$$

is completely determined by four real numbers. This remarkable property leads to the existence of the fundamental isomorphism between the space of the second-rank

JHEP07(1998)016

Hermitean spinors and the tangent space the elements of which (vectors) are also determined by four real numbers. As is known this isomorphism can be established by the "mixed quantity" (the set of generalized Pauli matrices) $\boldsymbol{\sigma}$ as

$$\boldsymbol{v} \equiv v^k \boldsymbol{e}_k \longrightarrow \boldsymbol{\sigma}(\boldsymbol{v},\ ,\ ) = v^k \boldsymbol{\sigma}(\boldsymbol{e}_k, \boldsymbol{\omega}^A, \boldsymbol{\omega}^{\dot B}) \boldsymbol{\omega}_A \otimes \boldsymbol{\omega}_{\dot B} \equiv v^k {\sigma_k}^{A\dot B} \boldsymbol{\omega}_A \otimes \boldsymbol{\omega}_{\dot B}\,. \tag{2.11}$$

Obviously the map ${\sigma_k}^{A\dot B}$ is Hermitean

$${\sigma_k}^{A\dot B} = {\bar\sigma_k}{}^{A\dot B} = {\sigma_k}^{\dot B A}\,. \tag{2.12}$$

In component representation the relation between two-component Hermitean spinor $\chi^{A\dot B}$ and vector v^k looks as follows

$$\chi^{A\dot B} = {\sigma_k}^{A\dot B} v^k\,. \tag{2.13}$$

The relation (2.13) must hold independently of the choice of a particular coordinate system. Thus the quantities ${\sigma_k}^{A\dot B}$ must transform as a covariant four-vector in respect to transformations of the vector space $T_x(M)$ and, independently, as a second-rank spinor in respect to spinor transformations.

One can arbitrarily identify the quadratic form $\chi_{A\dot B}\chi^{A\dot B}$ with the metrical quadratic form $g_{kl}v^k v^l$ using the isomorphic map ${\sigma_k}^{A\dot B}$:

$$g_{kl}v^k v^l = \varepsilon_{AC}\varepsilon_{\dot B\dot D}\chi^{A\dot B}\chi^{C\dot D} = \varepsilon_{AC}\varepsilon_{\dot B\dot D}{\sigma_k}^{A\dot B}{\sigma_l}^{C\dot D}v^k v^l \tag{2.14}$$

identically in v^k. Thus

$$g_{ik} = \varepsilon_{AC}\varepsilon_{\dot B\dot D}{\sigma_i}^{A\dot B}{\sigma_k}^{C\dot D}\,. \tag{2.15}$$

This equation shows that the metric tensor $\boldsymbol{g}$ is uniquely determined by the spinor metric $\boldsymbol{\varepsilon}$ and $\bar{\boldsymbol{\varepsilon}}$ through the map $\boldsymbol{\sigma}$. The above equation is equivalent to one of the following relations:

$${\sigma_k}^{A\dot C}\sigma^k{}_{B\dot D} = -\delta^A_B\delta^{\dot C}_{\dot D}\,; \tag{2.16}$$

$$\sigma^{kA\dot C}\sigma^i{}_{B\dot C} + \sigma^{iA\dot C}\sigma^k{}_{B\dot C} = g^{ik}\delta^A_B\,. \tag{2.17}$$

Eq. (2.15) holds in the tangent space where the metric $\boldsymbol{g}$ is taken to be Lorentzian. To go from local Lorentz metric g_{ik} to arbitrary "world" metric $g_{\mu\nu}$ one needs to introduce the tetrad fields h^i_μ:

$$g_{\mu\nu} = h^i_\mu h^k_\nu g_{ik}\,, \tag{2.18}$$

so that the "world" quantities ${\sigma_\mu}^{C\dot D}$ read:

$${\sigma_\mu}^{C\dot D} = h^k_\mu {\sigma_k}^{C\dot D} \tag{2.19}$$

and the main relation (2.15) becomes

$$g_{\mu\nu} = \varepsilon_{AC}\varepsilon_{\dot B\dot D}{\sigma_\mu}^{A\dot B}{\sigma_\nu}^{C\dot D}\,. \tag{2.20}$$

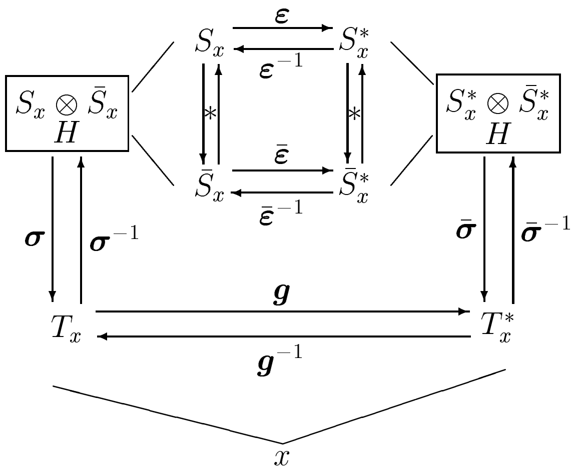

Figure 1: Relations between spinor spaces and tangent vector space at a single spacetime point x.

Let us consider now the group properties of the obtained relations. As we have seen the most general group acting in spinor space is $GL(2,\mathbb{C})$ (see Eq. (2.5)). However it is clear that ε_{AB} is left invariant only by unimodular transformations

$$|G^A_B| = 1 \tag{2.21}$$

constituting the subgroup $SL(2,\mathbb{C}) \subset GL(2,\mathbb{C})$. The last condition determines two of the eight real parameters specifying G^A_B. The remaining six parameters permit us to coordinate G^A_B to a six parameter Lorentz group $SO(1,3)$ in the following way. Let Λ^k_l be a Lorentz transformation. Then Λ^k_l and the spinor transformation $G^A_B \in SL(2,\mathbb{C})$ applied in concert leave the σ^{kA}_B invariant if

$$\sigma^{kA}{}_{\dot B} = \sigma'^{kA}{}_{\dot B} = \Lambda^k_l G^A_C \sigma^{lC}{}_{\dot D} G^{-1\dot D}{}_{\dot B}\,, \tag{2.22}$$

i.e. if

$$\Lambda^k_l \sigma^{lA}{}_{\dot B} = G^{-1A}{}_C \sigma^{kC}{}_{\dot D} G^{\dot D}_{\dot B}\,. \tag{2.23}$$

When the Lorentz transformation Λ^k_l is given these equations can be solved in a straightforward way. Thus there are exactly two spinor transformations corresponding to every Lorentz transformation which is represented at all spinor space, namely $\pm G^A_B$, where $G^A_B \in SL(2,\mathbb{C})$ is a unimodular solution of (2.23). Therefore the group $SL(2,\mathbb{C})$ is the double-covering of the Lorentz group $SO(1,3)$.

When we enlarge the group of transformations of spinor space to $GL(2,\mathbb{C})$ and the group of transformations of tangent vector space to $GL(4,\mathbb{R})$ then, as is easy to understand, there is no longer way to preserve both $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ invariant. In this case the quantities $\boldsymbol{\sigma}$ only establish the fundamental isomorphism given by Eq. (2.11).

So in general metric-affine space-time we lose the group relations (2.23). Nevertheless the main relation (2.15) between vectors and spinors holds.

Summarizing we can represent the relationships between four-vectors and two-component spinors at a single space-time point x as depicted in Fig. 1.

3. Bispinors and Generalized Dirac Matrices

Let us construct now at a point x a direct sum space $B_x \equiv S^*_x \oplus \bar{S}_x$ and represent its element Ψ by a column

$$\Psi \equiv \begin{pmatrix} \xi_A \\ \bar{\eta}^{\dot{A}} \end{pmatrix} . \tag{3.1}$$

The dual space B^*_x is then represented by $B^*_x = S_x \oplus \bar{S}^*_x$. It is convenient to represent its element $\tilde{\Psi}$ by a row vector

$$\tilde{\Psi} \equiv (\, \xi^A, \bar{\eta}_{\dot{A}} \,) . \tag{3.2}$$

Instead of dual spinors $\tilde{\Psi}$ it is more convenient to use the so-called adjoint spinors $\bar{\Psi}$:

$$\bar{\Psi} \equiv \Psi^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\, \eta^A, \bar{\xi}_{\dot{A}} \,) \in B^*_x . \tag{3.3}$$

The action of the group $GL(2,\mathbb{C})$ on two component spinors is given by the formulae (2.5) and (2.6). The corresponding representation matrix of $GL(2,\mathbb{C})$ in the space B_x is given by

$$G = \begin{pmatrix} G^{-1A}{}_B & 0 \\ 0 & \bar{G}^{\dot{A}}{}_{\dot{B}} \end{pmatrix} . \tag{3.4}$$

This matrix acts on the Ψ from the left. In the space B^*_x we have

$$G^{-1} = \begin{pmatrix} G^A{}_B & 0 \\ 0 & \bar{G}^{-1\dot{B}}{}_{\dot{A}} \end{pmatrix} , \tag{3.5}$$

which acts on $\bar{\Psi}$ from the right. An element Ξ of the tensor product space $B_x \otimes B^*_x$ has the following matrix form:

$$\Xi = \begin{pmatrix} A_A{}^B & B_{A\dot{B}} \\ & \\ C^{\dot{A}B} & D^{\dot{A}}{}_{\dot{B}} \end{pmatrix} . \tag{3.6}$$

For the basis of the tensor product space the following sixteen matrices are often used:

$$\begin{aligned} \mathbf{1} &\equiv \begin{pmatrix} \delta_A{}^B & 0 \\ 0 & \delta^{\dot{A}}{}_{\dot{B}} \end{pmatrix} ; \quad \gamma^5 \equiv \begin{pmatrix} \delta_A{}^B & 0 \\ 0 & -\delta^{\dot{A}}{}_{\dot{B}} \end{pmatrix} ; \\ \gamma^\mu &\equiv \sqrt{2} \begin{pmatrix} 0 & \sigma^\mu{}_{A\dot{B}} \\ \sigma^{\mu\dot{A}B} & 0 \end{pmatrix} ; \quad \gamma^5\gamma^\mu \equiv \sqrt{2} \begin{pmatrix} 0 & \sigma^\mu{}_{A\dot{B}} \\ -\sigma^{\mu\dot{A}B} & 0 \end{pmatrix} ; \end{aligned} \tag{3.7}$$

$$\sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{2} [\gamma^\mu, \gamma^\nu] . \tag{3.8}$$

The γ-matrices satisfy the well-known relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1} ; \tag{3.9}$$

$$\left\{\gamma^5, \gamma^\mu\right\} = 0 ; \tag{3.10}$$

$$\gamma^5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma . \tag{3.11}$$

It is instructive to note that the Dirac matrices γ^μ can exist in arbitrary metric-affine space independently of the existence of any underlying spinor structure on this manifold. The only relation between γ^μ and $g^{\mu\nu}$ (3.9) holds with no reference to any spinor structure. This feature of Dirac matrices up to now was not exploited effectively in physics.

4. Parallel Transport and Spinor Connection

So far our analysis was developed for separate points of space-time. At each point of the manifold we have the construct represented in Fig.1. To connect these constructs at various points to each other we should introduce a connection on the manifold, i.e. we should set a rule of parallel transport of spinors. To do this we should investigate how the spinor metric $\boldsymbol{\varepsilon}$ and the generalized Pauli matrices $\boldsymbol{\sigma}$ change under parallel transport. First let us remark that the quantities $\boldsymbol{g}$ and $\boldsymbol{\sigma}$ can be globally and continuously defined on the manifold through the continuous tetrad field $h^a_\mu(x)$:

$$\sigma_\mu{}^{A\dot{B}}(x) = h^i_\mu(x)\sigma_i{}^{A\dot{B}}\,; \tag{4.1}$$

$$g_{\mu\nu} = h^i_\mu(x)h^k_\nu(x)g_{ik}\,. \tag{4.2}$$

If there exists an orthonormal continuous tetrad on the manifold then, as Geroch has proved [10], it is possible to introduce an usual Cartan spinor structure over the manifold as $SL(2,\mathbb{C})$ principal fiber bundle. However the existence of a continuous orthonormal tetrad restricts the metric-affine manifold to Riemann-Cartan type. In the metric-affine manifold of general type there is no globally defined orthonormal tetrad, so the usual spinor structure as $SL(2,\mathbb{C})$ principal fiber bundle cannot be defined here. However it is possible to define a generalized spinor structure as a $GL(2,\mathbb{C})$ principal fiber bundle over the space-time manifold. For this aim we need (i) the existence of continuous tetrad of general type (which is assumed), and (ii) the notion of spinor connection correctly defined on metric-affine manifold. Moreover we assume that the spinor metric $\boldsymbol{\varepsilon}$ is given as a continuous field $\boldsymbol{\varepsilon}(x)$ on the manifold.

Let us define a linear connection on the above spinor bundle as a one-form with the values in the Lie algebra $gl(4,\mathbb{R})$. For a section of the bundle (the spinor field $\xi(x)$) this introduces the coefficients of a local connection $\Gamma_\mu(x)$. Under a change of a local basis given by the relations (2.5) they transform as

$$\Gamma_\mu \to \Gamma'_\mu = G\Gamma_\mu G^{-1} + G\partial_\mu G^{-1}\,, \tag{4.3}$$

where $G \in GL(2,\mathbb{C})$.

Using the connection Γ_μ one can define the covariant derivative of a spinor field $\xi(x)$ in the following way:

$$\nabla_\mu \xi^A{}_{\dot{B}} = \partial_\mu \xi^A{}_{\dot{B}} + \Gamma^A{}_{C\mu}\xi^C{}_{\dot{B}} - \Gamma^{\dot{D}}{}_{\dot{B}\mu}\xi^A{}_{\dot{D}}\,. \tag{4.4}$$

Let us calculate now the covariant derivative $\nabla_\mu \varepsilon_{AB}$ and $\nabla_\mu \sigma_\nu{}^{A\dot{B}}$. Since $\nabla_\mu \varepsilon_{AB}$ is antisymmetric in the indices A and B it is proportional to ε_{AB}. Therefore one can write

$$\nabla_\mu \varepsilon_{AB} = \partial_\mu \varepsilon_{AB} + 2\Gamma_{[AB]\mu} = \Phi_\mu \varepsilon_{AB} \tag{4.5}$$

where Φ_μ is a complex vector measuring the failure of spinor connection to preserve spinor metric under parallel transport. We call this vector the *nonmetricity vector*. Taking $\varepsilon_{AB}(x)$ in the most general form

$$\varepsilon_{AB}(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \varepsilon(x) \mathrm{e}^{\mathrm{i}\theta(x)} \tag{4.6}$$

where $\varepsilon(x)$ and $\theta(x)$ are real functions, we obtain from (4.5):

$$\Gamma^C{}_{C\mu} = \frac{1}{2}\varepsilon^{AB}\partial_\mu \varepsilon_{AB} - \Phi_\mu = \partial_\mu \ln \varepsilon + \mathrm{i}\partial_\mu \theta - \Phi_\mu \, . \tag{4.7}$$

From the equation complex conjugate to (4.5) one has:

$$\Gamma^{\dot{C}}{}_{\dot{C}\mu} = \frac{1}{2}\varepsilon^{\dot{A}\dot{B}}\partial_\mu \varepsilon_{\dot{A}\dot{B}} - \Phi^*_\mu = \partial_\mu \ln \varepsilon - \mathrm{i}\partial_\mu \theta - \Phi^*_\mu \, , \tag{4.8}$$

where Φ^*_μ denotes the complex conjugate of Φ_μ.

Now let us consider the covariant derivative of $\boldsymbol{\sigma}$. Since $\nabla_\rho \sigma_\mu{}^{A\dot{B}}$ is Hermitean in the indices $A\dot{B}$ it can be expressed through a linear combination of four independent Hermitean basis matrices $\sigma_\mu{}^{A\dot{B}}$ with real coefficients:

$$\nabla_\rho \sigma_\mu{}^{A\dot{B}} = N_{\rho\mu}{}^\lambda \sigma_\lambda{}^{A\dot{B}} \, . \tag{4.9}$$

The tensor $N_{\rho\mu}{}^\lambda$ will be called the *distortion tensor* since it measures the distortion of $\sigma_\mu{}^{A\dot{B}}$ under parallel transport. With the aid of (4.5) and (4.9) one can express the nonmetricity tensor (see Appendix A for main notations and definitions of most important quantities of the metric-affine space-time given in [1]) via Φ_μ and $N_{\rho\mu\nu}$:

$$W_{\rho\mu\nu} = 2g_{\mu\nu}\mathrm{Re}\Phi_\rho + 2N_{\rho(\mu\nu)} \, . \tag{4.10}$$

It is convenient to decompose $N_{\rho\mu\nu}$ as follows:

$$\begin{gathered} N_{\rho\mu\nu} = \tilde{N}_{\rho\mu\nu} + g_{\mu\nu} N_\rho \, ; \\ \tilde{N}_{\rho\lambda}{}^\lambda = 0 \, ; \quad N_\rho = \frac{1}{4} N_{\rho\lambda}{}^\lambda \, , \end{gathered} \tag{4.11}$$

so that the Weyl vector W_μ becomes:

$$W_\mu = \mathrm{Re}\Phi_\mu + N_\mu \, . \tag{4.12}$$

From the Eq.(4.9) after some calculations one obtains:

$$\Gamma^A{}_{B\mu} = \frac{1}{2}\left[\Sigma_{\lambda\rho\mu} + M_{\lambda\rho\mu} + \tilde{N}_{\mu[\rho\lambda]}\right]\sigma^{\lambda A\dot{C}}\sigma^\rho{}_{B\dot{C}} - \frac{1}{2}\delta^A{}_B\left(\Phi_\mu - \partial_\mu \ln \varepsilon - \mathrm{i}\partial_\mu\theta\right), \tag{4.13}$$

where

$$\Sigma_{\lambda\rho\mu} \stackrel{def}{\equiv} \sigma_{\lambda A\dot{B}}\partial_{[\rho}\sigma_{\mu]}{}^{A\dot{B}} + \sigma_{\mu A\dot{B}}\partial_{[\rho}\sigma_{\lambda]}{}^{A\dot{B}} + \sigma_{\rho A\dot{B}}\partial_{[\mu}\sigma_{\lambda]}{}^{A\dot{B}} + 2g_{\mu[\lambda}\partial_{\rho]}\ln\varepsilon \tag{4.14}$$

and $M_{\lambda\rho\mu}$ is the contortion tensor (see Appendix A).

With the aid of (4.7), (4.8) and (4.13) using definition (3.7) one can calculate the covariant derivative of the γ matrices:

$$\nabla_\rho\gamma^\mu = -N_{\rho\lambda}{}^{\mu}\gamma^\lambda - (1-\gamma^5)\gamma^\mu \mathrm{Re}\Phi_\rho\,. \tag{4.15}$$

For covariant derivative of the four-component spinor Ψ one has:

$$\nabla_\mu\Psi = \begin{pmatrix} \nabla_\mu\xi_A \\ \nabla_\mu\bar{\eta}^{\dot{A}} \end{pmatrix} = \partial_\mu\Psi - \Gamma_\mu\Psi\,, \tag{4.16}$$

where

$$\Gamma_\mu = \begin{pmatrix} \Gamma^A{}_{B\mu} & 0 \\ 0 & -\Gamma^{\dot{B}}{}_{\dot{A}\mu} \end{pmatrix}\,. \tag{4.17}$$

For the adjoint spinor $\bar{\Psi}$ one has:

$$\nabla_\mu\bar{\Psi} = \partial_\mu\bar{\Psi} + \bar{\Psi}\Gamma_\mu\,. \tag{4.18}$$

The explicit expression for Γ_μ is:

$$\Gamma_\mu = \frac{\mathrm{i}}{4}\sigma^{\alpha\beta}(\Sigma_{\alpha\beta\mu} + M_{\alpha\beta\mu} + \tilde{N}_{\mu\alpha\beta}) - \frac{1}{2}(\mathrm{Re}\Phi_\mu - \partial_\mu\ln\varepsilon)\gamma^5 - \frac{\mathrm{i}}{2}(\mathrm{Im}\Phi_\mu - \partial_\mu\theta)\,. \tag{4.19}$$

5. Dirac Equation

Now we are ready to write down the Dirac equation in the metric-affine space-time. The action for the four component spinor field $\Psi(x)$ is

$$S = \int \sqrt{-g}L(\Psi, \bar{\Psi}, g, \Gamma)d^4x\,, \tag{5.1}$$

where the Lagrangian L of the spinor field Ψ is taken in the form

$$L = \frac{\mathrm{i}}{2}\left[\bar{\Psi}\gamma^\mu\nabla_\mu\Psi - (\nabla_\mu\bar{\Psi})\gamma^\mu\Psi\right] - m\bar{\Psi}\Psi\,. \tag{5.2}$$

Using the explicit expression for spinor connection (4.19) one can represent the Lagrangian in the form:

$$\begin{aligned} L = \frac{\mathrm{i}}{2}\left[\bar{\Psi}\gamma^\mu\partial_\mu\Psi - (\partial_\mu\bar{\Psi})\gamma^\mu\Psi\right] - m\bar{\Psi}\Psi - \frac{1}{2}(\mathrm{Im}\Phi_\mu - \partial_\mu\theta)\bar{\Psi}\gamma^\mu\Psi \\ + \frac{1}{4}(\Sigma_{\alpha\beta\mu} + M_{\alpha\beta\mu} + \tilde{N}_{\alpha\beta\mu})\varepsilon^{\alpha\beta\mu\nu}\bar{\Psi}\gamma^5\gamma_\nu\Psi\,. \end{aligned} \tag{5.3}$$

Varying the action with this Lagrangian with respect to Ψ and $\bar{\Psi}$ one obtains the following equations:

$$\left[\mathrm{i}\gamma^\mu\partial_\mu + \frac{\mathrm{i}}{2\sqrt{-g}}\partial_\mu(\sqrt{-g}\gamma^\mu) + \frac{1}{4}(\Sigma_{\alpha\beta\mu} + M_{\alpha\beta\mu} + \tilde{N}_{\alpha\beta\mu})\varepsilon^{\alpha\beta\mu\nu}\gamma^5\gamma_\nu \right.$$
$$\left. -\frac{1}{2}(\mathrm{Im}\Phi_\mu - \partial_\mu\theta) - m\right]\Psi = 0; \qquad (5.4)$$

$$\bar{\Psi}\left[\mathrm{i}\gamma^\mu\overleftarrow{\partial} + \frac{\mathrm{i}}{2\sqrt{-g}}\partial_\mu(\sqrt{-g}\gamma^\mu) - \frac{1}{4}(\Sigma_{\alpha\beta\mu} + M_{\alpha\beta\mu} + \tilde{N}_{\alpha\beta\mu})\varepsilon^{\alpha\beta\mu\nu}\gamma^5\gamma_\nu \right.$$
$$\left. +\frac{1}{2}(\mathrm{Im}\Phi_\mu - \partial_\mu\theta) + m\right] = 0\,. \qquad (5.5)$$

From these equations we get the conservation law of probability:

$$\partial_\mu(\sqrt{-g}\bar{\Psi}\gamma^\mu\Psi) = 0\,. \qquad (5.6)$$

When nonmetricity vanishes, i.e. $\mathrm{Re}\Phi_\mu = \mathrm{Im}\Phi_\mu = N_{\alpha\beta\mu} = 0$, equations (5.4) and (5.5) coincide with the equations for Dirac field in Riemannian-Cartan space-time [18], and our spinor-type objects transform into the ordinary Cartan spinors. When both torsion and nonmetricity vanish Eqs. (5.4) and (5.5) reproduce the well-known Dirac equations in Riemannian space-time.

6. Physical considerations

Let us consider if generalized spinor-type fields can shed light on the physical origin of nonmetrical properties of space-time characterizing by $W_{\alpha\beta\mu}$, $N_{\alpha\beta\mu}$ and Φ_μ.

Since $M_{\alpha\beta\mu}$ enters equations (5.4) and (5.5) in the totally antisymmetric form, the nonmetricity tensor $W_{\alpha\beta\mu}$ drops from the equations. Therefore nonmetricity $W_{\alpha\beta\mu}$ does not couple with spinor fields as far as one adopts minimal coupling of matter with gravity, and hence generalized spinors cannot form a hypothetical hypermomentum fluid (the source of nonmetricity) discussed in the literature (see e.g., [13]). Furthermore neither $\mathrm{Re}\Phi_\mu$ nor N_μ enter these equations. Thus the Weyl vector W_μ does not couple directly with spinor fields as well. An interaction of spinor fields with the nonmetrical properties of space-time enters through the imaginary part of the nonmetrical vector, $\mathrm{Im}\Phi_\mu$, and through the antisymmetric part of the distortion tensor, $\tilde{N}_{\lambda[\mu\nu]}$.

The tensor $\tilde{N}_{\lambda[\mu\nu]}$ has the same symmetries as the torsion tensor. Since only the totally antisymmetric part of $\tilde{N}_{\lambda[\mu\nu]}$ enters (5.4) and (5.5), and it couples with the Dirac particles in the same way as the torsion tensor, the $\tilde{N}_{\lambda[\mu\nu]}$ is excited by the spin current of the Dirac particles. This term has purely fermionic origin. Really, consider the Proca field with the field strength $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} - S^\lambda{}_{\mu\nu}A_\lambda$. From the Lagrangian $L(A_\mu) = -(1/8\pi)(1/2F_{\mu\nu}F^{\mu\nu} - m^2A_\mu A^\mu)$ one can see that vector particles do not minimally couple with the $\tilde{N}_{\lambda[\mu\nu]}$. As is easy to check neither do scalar and tensor particles. Thus the quantity $\tilde{N}_{\lambda[\mu\nu]}$ may be called after Hayashi [16] a fermion number field. To discuss the physical reality of the $\tilde{N}_{\lambda[\mu\nu]}$ one needs to know its dynamics.

However this is not a simple problem for, as is known [13], the torsion tensor has many varieties of Lagrangians, and it is very difficult to find out a natural Lagrangian for the $\tilde{N}_{\lambda[\mu\nu]}$. So we leave the study of $\tilde{N}_{\lambda[\mu\nu]}$ for future investigations.

Let us turn now to the imaginary part of the nonmetrical vector, $\mathrm{Im}\Phi_\mu$. At first glance it couples with fermionic matter in a universal way and has a purely fermionic origin. Consider, however, an enlarged conformally covariant theory with the transformation group $GL(1,\mathbb{C})$. An element of the group $GL(1,\mathbb{C})$ transforms a spinor metric into another spinor metric. This group is a complexified version of the Weyl conformal transformation group $GL(1,\mathbb{R})$, whose element transforms a metric tensor into another metric tensor. This group was first introduced on the Weyl-Cartan space-time ($N_{\lambda\mu\nu} = 0$) in [16]. Obviously equations (5.4) and (5.5) are not manifestly covariant under the group $GL(1,\mathbb{C})$. One can rewrite them in manifestly covariant form by introducing a $GL(4,\mathbb{R}) \times GL(2,\mathbb{C}) \times GL(1,\mathbb{C})$ covariant differentiation called the complex cocovariant differentiation. If one provides the physical fields by a complex conformal weights then one obtains a possibility to identify the imaginary part of the nonmetrical vector, $\mathrm{Im}\Phi_\mu$, with the electromagnetic field. In such an approach the $\mathrm{Im}\Phi_\mu$ does not universally couple with fermionic matter. It couples only with the fields possessing a nonzero imaginary part of the complex conformal weight which can be reasonable identified with the electric charge. Moreover, in conformally covariant theory the Weyl vector (the real part of the nonmetrical vector) universally couples with physical fields through the mass factor m which is not a constant number but a coscalar of power -1. Therefore one obtains an intriguing possibility of unification of the electromagnetic field and particle masses in the single complex nonmetrical vector Φ_μ. A detailed study of the complex conformally covariant theory including such a geometrization of the electromagnetic field and mass generation will be done in a separate paper.

7. Conclusion

In this paper we have constructed a generalized spinor structure over the metric-affine manifold as a $GL(2,\mathbb{C})$ principal fiber bundle. Spinor fields in this construction differ from the ordinary spinors in respect that the former do not carry an irreducible representation of the orthogonal Lorentz group. The presence of proper nonmetricity destroys the usual notion of spinor.

Riemannian space-time is characterized by a symmetric and metric compatible connection, i.e. by the restriction:

$$\nabla_\rho g_{\mu\nu} = 0\,. \tag{7.1}$$

This condition is crucial for Riemannian space-time and must be taken into account in spinor formalism. In two-component spinor theory this single condition is usually replaced by two other conditions:

$$\nabla_\rho \sigma_\mu{}^{A\dot{B}} = 0\,; \tag{7.2}$$

$$\nabla_\rho (\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}}) = 0\,. \tag{7.3}$$

The first condition states that the connection preserves the linear isomorphism $\boldsymbol{\sigma}$ and the second one states that the connection preserves any volume in spinor space. As is easy to see these two conditions are enough for two-component spinors to carry an irreducible representation of the Lorentz group.

Solving (7.2) and (7.3) for the spinor connection one can see that these conditions fix $\Gamma_{(AB)\mu}$ and $\mathrm{Re}\Gamma^C{}_{C\mu}$ but $\mathrm{Im}\Gamma^C{}_{C\mu}$ remains indefinite. Let us remark that in bispinor theory the metric condition (7.1) has been often replaced by the condition

$$\nabla_\mu \gamma_\nu = 0\,. \tag{7.4}$$

Eq.(7.4) fix the spinor connection Γ_μ up to the trace $\mathrm{Tr}\Gamma_\mu$ which remains indefinite. This indefiniteness is nothing but the indefiniteness of $\mathrm{Im}\Gamma^C{}_{C\mu}$. As is well known this indefiniteness can be eliminated by replacing (7.3) by a stronger condition

$$\nabla_\mu \varepsilon_{AB} = 0\,, \tag{7.5}$$

which states that the connection preserves the inner product in the spinor space. However it is instructive to note that neither of the conditions (7.2)-(7.5) follow from (7.1) directly. All of them are additional constraints imposed on the fundamental spinor objects $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$.

In metric-affine space-time the metricity condition (7.1) is relaxed and two new objects $N_{\alpha\beta\mu}$ and Φ_μ come into play. The symmetric part of the distortion tensor $N_{\alpha(\beta\mu)}$ is directly related with the nonmetricity tensor $W_{\alpha\beta\mu}$ via relation (4.10) but the other part $N_{\alpha[\beta\mu]}$ remains indefinite. Analogously only real part of the nonmetrical vector Φ_μ is connected with the Weyl vector through the relation (4.12). The imaginary part of Φ_μ remains indefinite. On the other hand just $N_{\alpha[\beta\mu]}$ and $\mathrm{Im}\Phi_\mu$ couple with fermionic matter and therefore should have a direct physical meaning. One of the possible interpretations was discussed in brief in the previous section. We remain this issue, however, for further investigations.

A. Basic relations in metric-affine space-time

Here we represent the main notations and some formulae concerning metric-affine space-time from the paper [1]. For simplicity all the formulae are written in holonomic coordinates.

The torsion tensor:

$$S^\mu{}_{\nu\rho} \stackrel{def}{\equiv} \frac{1}{2}(\Gamma^\mu{}_{\nu\rho} - \Gamma^\mu{}_{\rho\nu})\,. \tag{A.1}$$

The torsion vector:

$$S_\mu \stackrel{def}{\equiv} S^\lambda{}_{\mu\lambda}\,. \tag{A.2}$$

The nonmetricity tensor:

$$W_{\rho\mu\nu} \stackrel{def}{\equiv} \nabla_\rho g_{\mu\nu}\,; \tag{A.3}$$

$$W^{\mu\nu}_\rho = -\nabla_\rho g^{\mu\nu}\,. \tag{A.4}$$

The Weyl vector:

$$W_\lambda \stackrel{def}{\equiv} \frac{1}{8} W_{\lambda\rho}{}^{\rho} \,. \tag{A.5}$$

The proper nonmetricity tensor:

$$\bar{W}_{\lambda\mu\nu} \stackrel{def}{\equiv} W_{\lambda\mu\nu} - 2W_\lambda g_{\mu\nu} \,; \tag{A.6}$$

$$\bar{W}_{\lambda\rho}{}^{\rho} = 0. \tag{A.7}$$

The covariant derivative of a tensor:

$$\nabla_\rho A^\mu{}_\nu = \partial_\rho A^\mu{}_\nu + \Gamma^\mu{}_{\lambda\rho} A^\lambda{}_\nu - \Gamma^\lambda{}_{\nu\rho} A^\mu{}_\lambda \,. \tag{A.8}$$

The full connection of the metric-affine space-time:

$$\Gamma^\mu{}_{\nu\rho} = \{{}_\nu{}^\mu{}_\rho\} + B^\mu{}_{\nu\rho} \,, \tag{A.9}$$

where $\{{}_\nu{}^\mu{}_\rho\}$ is the Christoffel connection

$$\left\{{}_\nu{}^\mu{}_\rho\right\} = \frac{1}{2} g^{\mu\lambda} (\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho}) \,, \tag{A.10}$$

and $B^\mu{}_{\nu\rho}$ is the deviation tensor:

$$B^\mu{}_{\nu\rho} = S^\mu{}_{\nu\rho} - S_\nu{}^\mu{}_\rho - S_\rho{}^\mu{}_\nu + \frac{1}{2} (W^\mu{}_{\nu\rho} - W_\nu{}^\mu{}_\rho - W_\rho{}^\mu{}_\nu) \,. \tag{A.11}$$

The contortion tensor:

$$M_{\mu\nu\rho} \stackrel{def}{\equiv} B_{[\mu\nu]\rho} = S_{\mu\nu\rho} - S_{\nu\mu\rho} - S_{\rho\mu\nu} + \bar{W}_{[\mu\nu]\rho} + 2W_{[\mu} g_{\nu]\rho} \,. \tag{A.12}$$

References

[1] E.A. Poberii, *Gen. Rel. Grav.,* **26** (1994) 1011.

[2] E.A. Poberii, in: *Proceedings of the Second Alexander Friedmann Seminar on Gravitation and Cosmology, 1994*, A. Grib, Yu.N. Gnedin and V.M. Mostepanenko eds., Friedmann Laboratory Publishing, St. Petersburg; *Helv. Phys. Acta* **67** (1994) 745.

[3] E. Cartan, *Leçons sur la Théorie des Spineurs.* Hermann, Paris 1938.

[4] V. Fock and D. Iwanenko, *Comp. Rend. Acad. Sci. Paris* **188** (1929) 1470.

[5] W.L. Bade and H. Jehle, *Rev. Mod. Phys.* **25** (1953) 714.

[6] P. Bergmann, *Phys. Rev.* **107** (1957) 625.

[7] R. Penrose, *Ann. Phys. (NY)* **10** (1962) 171.

[8] A. Peres, *Suppl. Nuovo Cim.* **24** (1962) 389.

[9] E.M. Corson, *Introduction to Tensors, Spinors and Relativistic Wave equations*, Hafner, New York 1953.

[10] R. Geroch, *J. Math. Phys.* **9** (1968) 1739.

[11] J. Plebanski, *Lectures on Nonlinear Electrodynamics*, Nordita 1972.

[12] Y. Ne'eman, *Found. Phys.* **13** (1983) 467.

[13] F.W. Hehl, J.D. McCrea, E.W. Mielke and Y. Ne'eman, *Phys. Rep.* **258** (1995) 1 [gr-qc/9402012].

[14] A.A. Grib, S.G. Mamayev and V.M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields*, Friedmann Laboratory Publishing, St.Petersburg 1988;
N.D. Birrell and P.C.W. Davies, (1982). *Quantum fields in curved space*, Cambridge University Press 1982.

[15] R. Adler, *J. Math. Phys.* **11** (1970) 1185.

[16] K.Hayashi and T. Shirafuji, *Prog. Theor. Phys.* **57** (1977) 302.

[17] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Interscience, New-York 1963.

[18] B. Datta, *Nuovo Cim.* **6B** (1976) 1; *Nuovo Cim.* **6B** (1976) 16.